Review of guiding center motion

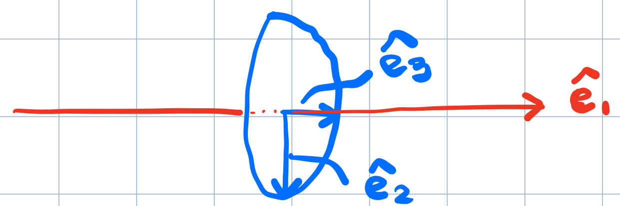
- single-particle motion in uniform field
- guiding center Lagrangian
  - ordering assumptions
  - variable transformations
  - gyro-averaging operation
- features of GC motion
  - energy & μ conservation
  - particle trapping
  - drifts & implications for toroidal confinement

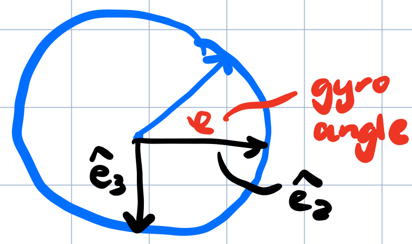
Single particle motion in a uniform field (Sec. 4.1)

Newton: $$m\ddot{\vec{r}} = q(\dot{\vec{r}} \times \vec{B})$$

- straight, uniform B: $\vec{B} = B\hat{e}_1$
- orthonormal basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$
- write velocity in "cylindrical" coordinates

$$\dot{\vec{r}}(t) = \underbrace{v_\parallel \hat{e}_1}_{\text{parallel}} + v_\perp \underbrace{(\cos(\varphi)\hat{e}_2 - \sin(\varphi)\hat{e}_3)}_{\perp \text{ rotation} \quad \vec{v}_\perp}$$





$$m\ddot{\vec r} = m v_\perp \dot\varphi\,(-\sin\varphi\,\hat e_2 - \cos\varphi\,\hat e_3)$$
$$= m\dot\varphi\,(\vec v_\perp \times \hat e_1)$$

Newton:

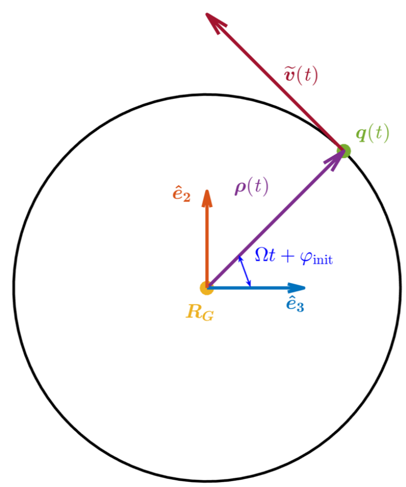
$$= \underbrace{q\,\dot{\vec r}\times\vec B}_{B q\,\vec v_\perp \times \hat e_1} = m\dot\varphi\,(\vec v_\perp \times \hat e_1)$$

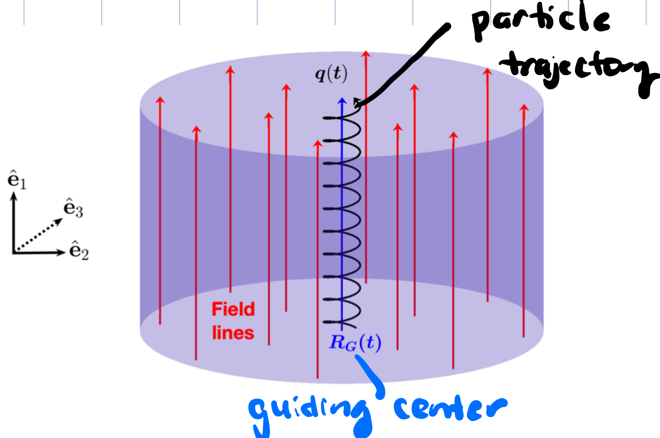
$$\dot\varphi = qB/m$$

$$\Omega = qB/m \qquad \text{gyro frequency}$$

solution: $r(t) = \underbrace{v_\parallel t\,\hat e_1}_{\text{const } \parallel \text{ velocity}} + \rho\,(\sin\varphi\,\hat e_2 + \cos\varphi\,\hat e_3)$

$$\underbrace{\rho = v_\perp/\Omega \quad \text{gyroradius}}_{\perp \text{ rotation}}$$





Conclusions:

- if $\rho \ll 1$, motion is "mostly" along field lines
- if $\Omega \gg v_\parallel / L$, we can "average" over fast gyration

Guiding Center Lagrangian (4.2)

- starting w/ charged-particle Lagrangian, we will build on observations from straight magnetic field case
- order terms in Lagrangian wrt assumption of strong magnetization
- average over fast gyration
- advantage: symmetries apparent

recall: $L(q,\dot{q}) = p(q,\dot{q})\cdot\dot{q} - H(q,p(q,\dot{q}))$

- We will use "phase-space Lagrangian" $\rightarrow$ more freedom in coordinate transform

$$L_{ph}(q,\dot{q},p,\dot{p}) = \underline{p\cdot\dot{q}} - \underline{H(q,p)}$$

where $\vec{p} = \partial L/\partial\dot{\vec{q}}$ & $H = \vec{p}\cdot\dot{\vec{q}} - L$

E-L eqns. :

$$\frac{d}{dt}\left[\frac{\partial L_{ph}}{\partial \dot{q}}\right] = \frac{\partial L}{\partial q} \quad ①$$

$$\frac{d}{dt}\left[\frac{\partial L_{ph}}{\partial \dot{p}}\right] = \frac{\partial L}{\partial p} \quad ②$$

① $$\frac{d}{dt}[p] = -\frac{\partial H}{\partial q} \rightarrow \dot{p} = -\frac{\partial H}{\partial q} \checkmark$$

② $$\frac{d}{dt}[0] = \dot{q} - \frac{\partial H}{\partial p} \rightarrow \dot{q} = \frac{\partial H}{\partial p} \checkmark$$

Charged particle motion

$$H(q,p) = \frac{|\vec{p} - q\vec{A}|^2}{2m} + q\Phi$$

$$L_{ph}(q,\dot{q},p,\dot{p}) = p\cdot\dot{q} - \frac{|\vec{p} - q\vec{A}|^2}{2m} - q\Phi$$

Guiding center assumptions

- L = equilibrium length scale (e.g. $L^{-1} \sim |\nabla B|/B$)
- ω = equilibrium time scale (e.g. $\omega \sim v_t/L$)

small gyroradius : $\rho/L \ll 1$
fast gyration : $\omega/\Omega \ll 1$

- define small parameter

$$\epsilon \sim \rho/L \sim \omega/\Omega \ll 1$$

- as in case of uniform field, expect "slow" timescale describes averaged guiding center motion

- fast timescale describes periodic perpendicular motion

Scale separation in stellarators

Name	Parameter	W7-AS [205]	LHD [161]	W7-X [282]
Electron Debye length	$\lambda_{D,e}$ [m]	3×10^{-5}	2×10^{-5}	9×10^{-5}
Ion gyroradius	ρ_i [m]	2×10^{-3}	3×10^{-3}	2×10^{-3}
Device minor radius	a [m]	0.20	0.60	0.50
Ion gyrofrequency	Ω_i [s^{-1}]	9×10^{7}	1×10^{8}	2×10^{8}
Collision frequency*	ν_{ee} [s^{-1}]	1×10^{5}	2×10^{5}	4×10^{3}
Energy confinement time	τ_E [s]	0.5	0.33	0.1

Guiding center coordinates

- $\hat{e}_1(\vec{r}) = \hat{b}(\vec{r}) = \vec{B}/|B|$ → non-uniform
- $\hat{e}_2(\vec{r}), \hat{e}_3(r)$ forms orthonormal basis

- decompose position:

$$\vec{r} = \underbrace{\vec{R}}_{\text{averaged GC motion}} + \underbrace{\vec{\rho}}_{\text{fast, periodic motion}}$$

$$\vec{\rho}(\rho, \varphi, \vec{R}) = \rho\,(\sin\varphi\,\hat{e}_2(\vec{R}) + \cos\varphi\,\hat{e}_3(\vec{R}))$$

- decompose velocity:

$$\vec{v}(v_\parallel, \dot{\varphi}, \rho, \vec{R}) = \underbrace{v_\parallel\,\hat{e}_1}_{\parallel\ \text{motion}} + \underbrace{\dot{\varphi}\,\vec{\rho} \times \hat{e}_1}_{\vec{v}_\perp}$$

$$\frac{(\vec{p} - q\vec{A})}{m} = \vec{v}$$

old coordinates : $\vec{q}, \vec{p}$ (+ derivatives)

$\Downarrow \vec{v}$

New coordinates : $\vec{R}, \rho, \varphi, v_{\parallel}$ (+ derivatives)

$$(\vec{r}, \dot{\vec{r}}, \vec{p}, \dot{\vec{p}}) \rightarrow (\vec{r}, \dot{\vec{r}}, \vec{v}, \dot{\vec{v}}) \rightarrow$$
$$(\vec{R}, \dot{\vec{R}}, \varphi, \dot{\varphi}, v_{\parallel}, \dot{v}_{\parallel}, \rho, \dot{\rho})$$

- resulting Lagrangian is "gyro-averaged"

$$\mathcal{L} = \langle L \rangle_{\varphi} = \int_0^{2\pi} d\varphi \, L \Big/ 2\pi$$

result:

$$\mathcal{L}(R, \dot{R}, \rho, \dot{\rho}, v_{\parallel}, \dot{v}_{\parallel}, \dot{\varphi}) =$$
$$\left(m v_{\parallel} \hat{b}(\vec{R}) + q\vec{A}(\vec{R})\right) \cdot \dot{\vec{R}} - \frac{m v_{\parallel}^2}{2}$$
$$+ \frac{\rho^2 (m\dot{\varphi}^2 - q\,\dot{\varphi}\, B(\vec{R}))}{2} - q\Phi(\vec{R})$$

Euler-Lagrange equations

$$\frac{d}{dt}\cancel{\left[\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right]} = \frac{\partial \mathcal{L}}{\partial \rho} \rightarrow \qquad \rho\dot{\varphi}(m\dot{\varphi} - qB) = 0$$

$$\dot{\varphi} = qB/m$$

$$\boxed{\Omega = qB/m}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right] = \cancel{\frac{\partial \mathcal{L}}{\partial \varphi}} \rightarrow \frac{d}{dt}\left[\frac{\rho^2(2m\dot{\varphi} - qB)}{2}\right] = 0$$

$$\rho^2(2m\dot{\varphi} - qB) = \rho^2 m\Omega$$

define $v_\perp = \rho\dot{\varphi} = \rho\Omega$

$$\mu = \frac{q\rho^2\Omega}{2} = \frac{m v_\perp^2}{2B} = \text{const.}$$

"magnetic moment"

$\rightarrow$ adiabatic invariant

$$\cancel{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{v}_\parallel}\right]} = \frac{\partial \mathcal{L}}{\partial v_\parallel} \rightarrow m\hat{b}\cdot\dot{\vec{R}} - m v_\parallel = 0$$

$$\boxed{v_\parallel = \hat{b}\cdot\dot{\vec{R}}}$$

parallel component of g.c. velocity

Energy conservation

- since $\mathcal{L}$ is time-indpt., $H = E$ is conserved

$$H = \frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}}\cdot\dot{\vec{R}} + \frac{\partial \mathcal{L}}{\partial \dot{\rho}}\dot{\rho} + \frac{\partial \mathcal{L}}{\partial \dot{v}_\parallel}\dot{v}_\parallel + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\dot{\varphi} - \mathcal{L}$$

$$\boxed{E = \underbrace{\frac{m v_\parallel^2}{2}}_{\text{parallel KE}} + \underbrace{\mu B(\vec{R})}_{\perp\ \text{KE}} + \underbrace{q\Phi(\vec{R})}_{\text{potential}}}$$

$$\mu = \frac{m v_\perp^2}{2B}$$

→ μ conservation provides "effective potential" for $v_\parallel$

Particle trapping

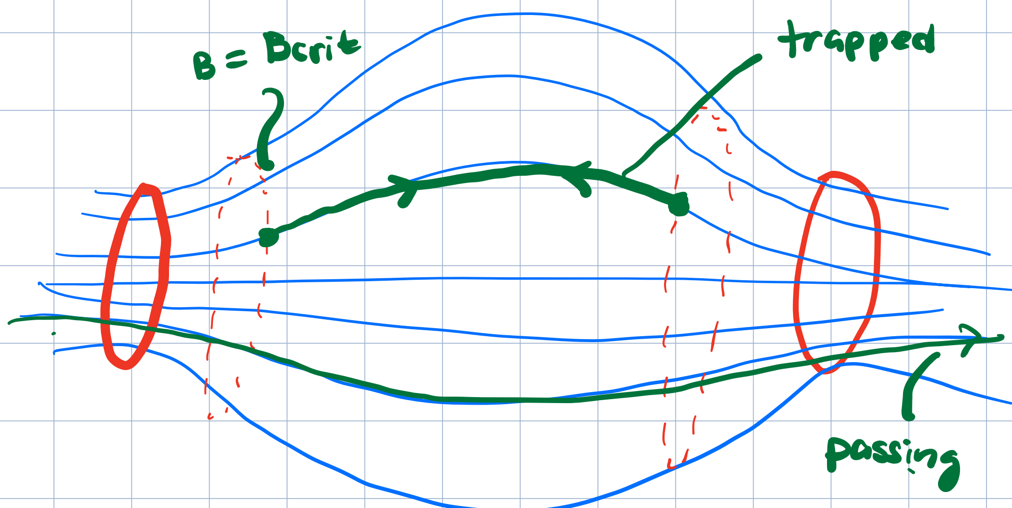
- E & μ conservation leads to particle trapping (ignore Φ for now)

$$v_{\parallel}^2 = \frac{2E - \mu B}{m} = 0 \quad \text{when}$$

$$B = B_{crit} = E/\mu$$

- if $B \geq B_{crit}$, then particle will mirror ($v_{\parallel}$ turns around) → "trapped particle"

- Otherwise, $v_{\parallel} \neq 0$ → "passing particle"



- On top of parallel motion w/ trapping effects, $\perp$ motion provides <u>drifts across field lines</u>

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{R}}\right] = \frac{\partial \mathcal{L}}{\partial R}$$

$$\downarrow$$

$$m \dot{v}_{\parallel} \hat{b} = \dot{R} \times \left(m v_{\parallel} \nabla \times \hat{b} + q\vec{B}\right) - \mu \nabla B$$

- parallel component, $\hat{b} \cdot (\quad\quad)$,

$$m\dot{v}_{\parallel} = m v_{\parallel} \quad \dot{R} \cdot \left[\cancel{\underbrace{(\nabla \times \hat{b}) \times \hat{b}}_{-\vec{\kappa}}}\right] - \mu \hat{b} \cdot \nabla B$$

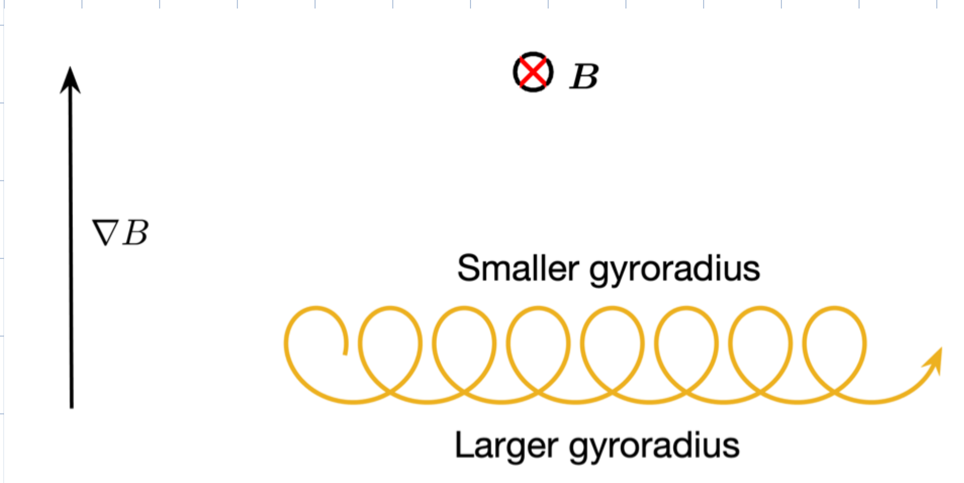
$$m \dot{v}_{\parallel} = -\mu \hat{b} \cdot \nabla B$$ "mirror force"

- perpendicular component: $\hat{b} \times (\quad)$,

$$\dot{R}_\perp = \underbrace{v_\parallel^2 \frac{\hat{b} \times \vec{K}}{\Omega}}_{\text{curvature drift}} + \underbrace{\frac{\mu}{q} \frac{\hat{b} \times \nabla B}{B}}_{\nabla B \text{ drift}} + \underbrace{\frac{\vec{E} \times \vec{B}}{B^2}}_{E \times B \text{ drift}}$$

$\vec{K} = \hat{b} \cdot \nabla \hat{b}$

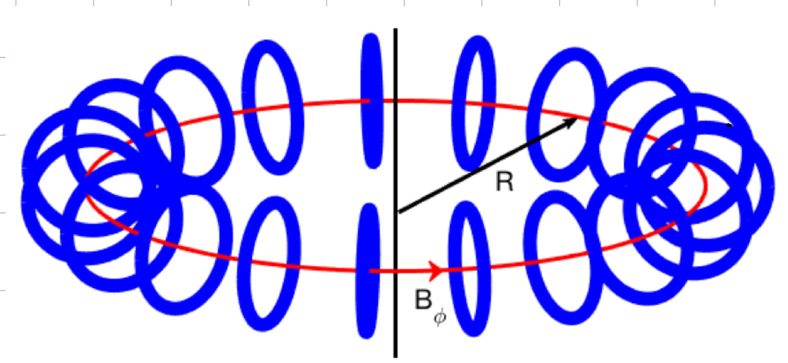
- curvature + ∇B drift known as magnetic drifts $\rightarrow$ depend on sign of charge
- $E \times B$ drift is indpt. of charge



Implications for toroidal confinement

- Cylindrical coordinates : (R, ϕ, Z)
- Suppose purely toroidal field :

$$\vec{B} = B_\phi \hat{\phi}$$



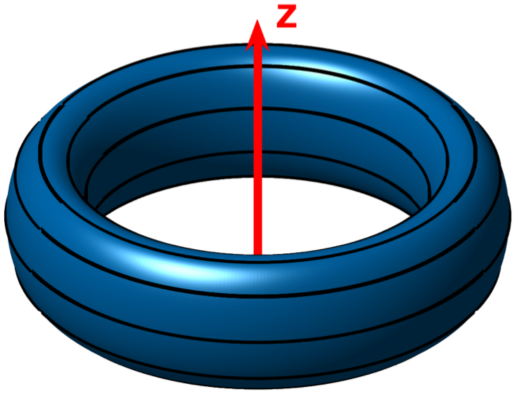
- curl-free condition : $\vec{J} = 0 \rightarrow \nabla \times \vec{B} = 0$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R B_\phi\right) = 0 \quad \rightarrow \quad B_\phi \sim 1/R$$

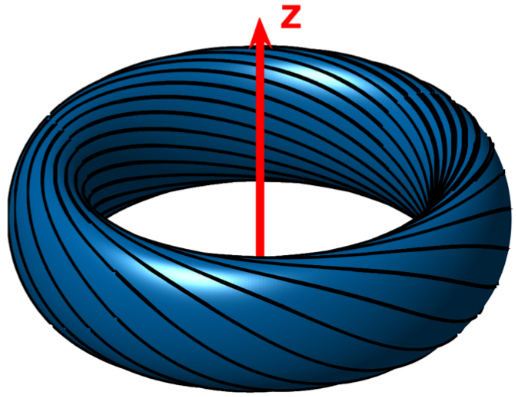
- ∇B drift $\propto \vec{B} \times \nabla B \propto \hat{\phi} \times (-\hat{R}) = \hat{z}$

- $\vec{R}$ drift $\propto \vec{B} \times \vec{R} \propto \hat{z}$

$\rightarrow$ unconfined drifts



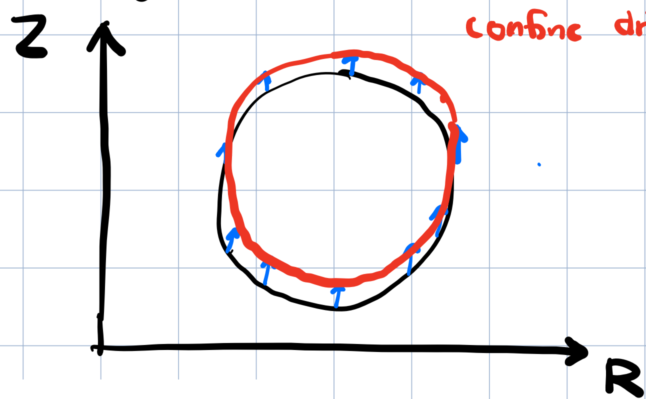
toroidal

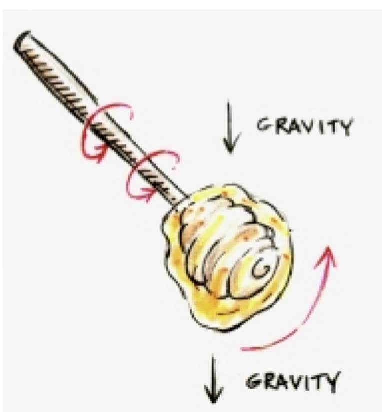


toroidal + poloidal

- poloidal component enables drift averaging

confine drifts!





analogous to honey dipper effect → rotation prevents deconfinement due to gravity

- although poloidal field is necessary for stellarator confinement, it is not sufficient due to p_ϕ non-conservation

→ need for "hidden symmetry"

example: QH configuration w/ symmetry breaking